J. Myanmar Acad. Arts Sci. 2018 **Vol. XVI. No.3**

A STUDY ON FILTERS

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Abstract

In this paper, firstly the classification of filters is discussed. Topological space plays a crucial role in this discussion. After that the relations between them are studied. Especially, the characterizations of ultrafilters are presented with detail proofs. Ultrafilter is a powerful tool both in set theory and in topology. Moreover, the comparisons of filters are expressed and some notions of filter basis and trace of filter are described. Finally, ultrafilter convergence theorem and convergence of Cauchy filter in topological vector space are investigated.

1. Some types of filters

1.1 Definitions

A collection $\mathcal{F}$ of subsets of a set X is called a **filter** on X if it satisfies the following axioms:

(F1) If A$\subset$ X and A contains a set B $\in \mathcal{F}$, then A $\in \mathcal{F}$.

(F2) The intersection of a finite collection of sets in $\mathcal{F}$ belongs to $\mathcal{F}$.

(F3) The empty subset of X does not belong to $\mathcal{F}$.

First let us examine a few elementary consequences of this definition. It follows from (F1) that X is a member of any filter on X.

Note that P(X) a collection of subsets of X is not a filter on X. However, it satisfies (F1) and (F2). Therefore it is sometimes called the improper filter on X. Conversely, if $\mathcal{F}$ is a collection of subsets of X containing the empty set and satisfying (F1) and (F2), it follows from (F1) that $\mathcal{F}$ = P(X), that is, $\mathcal{F}$ is improper.

Let X be a set and $\mathcal{A} \subset$ P(X) a collection of subsets. Then $\mathcal{A}$ has the **finite intersection property** (FIP) if any finite intersection of sets in$\mathcal{A}$ is non-empty. (From the axioms (F2) and (F3) that a filter has the FIP.)

The **cofinite filter** on an infinite set X is the set of all subsets A of X such that the complement of A in X is finite.

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That is, $\mathcal{F} = \{A \subseteq X: X \setminus A \text{ is finite }\}$. (or)

$\mathcal{F} = \{X \setminus A : A \subseteq X \text{ is finite }\}$.

This filter on $X = \mathbb{N}$, the set of natural numbers, is also called the **Fréchet filter**on $\mathbb{N}$.

A maximal element of the set of all filters on X is called an **ultrafilter** on X.

For any non-empty subset M of X, the set $\{A \subseteq X : M \subseteq A\}$ is a filter on X, the **principal filter** generated by M.

For any a $\in$ X the set $\{ A \subseteq X : a \in A \}$ is the **principal ultrafilter** defined by a.

Any ultrafilter that is not principal is called **non-principal ultrafilter**.

A filter $\mathcal{F}$ on X is**free** if the intersection of all sets in $\mathcal{F}$ is empty. That is, $\cap_{A \in \mathcal{F}} A = \emptyset$.

Let X be a set and $\mathcal{A} \subset P(X)$ a collection of subsets. The (im)proper filter generated by $\mathcal{A}$ is the set

$<\boldsymbol{\mathcal{A}}> = \cap(\{ \mathcal{F} \subset P(X) : \mathcal{F} \supset \mathcal{A}$ and $\mathcal{F}$ is a(n) (im) proper filter on X$\}$).

So $<\mathcal{A}>$ is the intersection of all (im) proper filters on X that contains set $\mathcal{A}$.

1.2 Example

The set of all neighborhoods of a point x $\in$ X is filter $\mathcal{B}(x)$ called the neighborhood filter of x.

2.Characterizations of the Ultrafilters

2.1 Theorem (Ultrafilter lemma)

Let X be a set and suppose $\mathcal{A} \subset P(X)$ has FIP. Then there is an ultrafilter $\mathcal{U}$ on X which contains all of $\mathcal{A}$.

Proof

Let the set $\mathfrak{B}$ consisting of all proper filters on X containing $\mathcal{A}$, partially ordered by set inclusion. Then $\mathfrak{B}$ is non-empty because $<\mathcal{A}> \in \mathfrak{B}$. Let $\mathcal{C}$ be a chain in $\mathfrak{B}$.

We will prove $\cup\,\mathcal{C} \in \mathfrak{B}$. For (F3), since any element of $\mathcal{C}$ does not contain empty set, $\emptyset \notin \cup\,\mathcal{C}$.For (F2), if A, B$\in \cup\,\mathcal{C}$, then there are C, D $\in \mathcal{C}$ such that A$\in$ C and B$\in$ D. Since $\mathcal{C}$ is a chain, we have $C \subset D$ without loss of generality. Consequently, A, B are elements of D and since D is a filter $A \cap B \in D \in \cup\,\mathcal{C}$. For (F1), it is a trivial matter to verify that $\cup\,\mathcal{C}$ is closed under supersets, so we have $\cup\,\mathcal{C} \in \mathfrak{B}$ indeed. This union is an upper bound of $\mathcal{C}$ in $\mathfrak{B}$. According to Zorn's lemma, $\mathfrak{B}$ has maximal elements. Let $\mathcal{U}$ be a maximal element of $\mathfrak{B}$. If $\mathcal{F} \supset \mathcal{U}$ is a filter, then $\mathcal{A} \subset \mathcal{F}$.By the maximality of $\mathcal{U}$, $\mathcal{F} \subset \mathcal{U}$ and we have $\mathcal{U} = \mathcal{F}$.So $\mathcal{U}$ is an ultrafilter and it contains all of $\mathcal{A}$.

2.2 Lemma

Let $A_1, A_2, \ldots, A_n \in$ P(X) such that $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{U}$ where$\mathcal{U}$ is an ultrafilter on X. Then $A_i \in \mathcal{U}$ for at least one i. In addition, if the sets are mutually disjoint, then $A_i \in \mathcal{U}$ for exactly one i.

Proof

Let $A_1 \cup A_2 \in \mathcal{U}$. Suppose (to the contrary) that neither $A_1 \in \mathcal{U}$ nor $A_2 \in \mathcal{U}$. Consider $\mathcal{M} = \{Z \in P(X) : A_1 \cup Z \in \mathcal{U}\}$.

First we need to show that $\mathcal{M}$ is a filter on X.For (F3), if $\emptyset$ is a member of $\mathcal{M}$, then $A_1 = A_1 \cup \emptyset \in \mathcal{U}$, contradiction. So $\emptyset \notin \mathcal{M}$.

For (F2), if $B_1, B_2 \in \mathcal{M}$, then $A_1 \cup B_1 \in \mathcal{U}$ and $A_1 \cup B_2 \in \mathcal{U}$. Now $(A_1 \cup B_1) \cap (A_1 \cup B_2) \in \mathcal{U}$ because $\mathcal{U}$ is a filter.That is, $A_1 \cup (B_1 \cap B_2) \in \mathcal{U}$.It follows that$B_1 \cap B_2 \in \mathcal{M}$.

For (F1), let V$\in$ P(X), $U \subset V$ and $U \in \mathcal{M}$. Then $A_1 \cup U \in \mathcal{U}$. Since $U \subset V$, $A_1 \cup U \subset A_1 \cup V$.Thus $A_1 \cup V \in \mathcal{U}$ because $\mathcal{U}$ is a filter. So $V \in \mathcal{M}$.

Therefore $\mathcal{M}$ is a filter on X.Moreover, we have $\mathcal{U} \subseteq \mathcal{M}$. Also, $\mathcal{U} \subsetneq \mathcal{M}$because $A_2 \in \mathcal{M} \backslash \mathcal{U}$, contradicting the maximality of $\mathcal{U}$.Our assumption is false, so $A_i \in \mathcal{U}$ for at least one i.Finally, if $A_1 \cap A_2 = \emptyset$ and $A_1, A_2 \in \mathcal{U}$ then this implies that $\emptyset \in \mathcal{U}$, a contradiction.The generalization to n≥2 follows by induction.

2.3 Theorem

Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is an ultrafilter if and only if for every $A \subset X$ either $A \in \mathcal{F}$or $X \setminus A \in \mathcal{F}$.

Proof

Suppose $\mathcal{F}$ is an ultrafilter.Let A ∈ P(X).The previous lemma holds since $A \cup (X \setminus A) = X \in \mathcal{F}$ and $A \cap (X \setminus A) = \emptyset$.

Conversely, suppose (to the contrary)that $\mathcal{F}$ is not an ultrafilter. Then there exists a filter $\mathcal{M}$ such that $\mathcal{F} \subsetneq \mathcal{M}$ and take $A \in \mathcal{M} \setminus \mathcal{F}$.

Thus $A \in \mathcal{M}$ and $A \notin \mathcal{F}$. So $X \setminus A \in \mathcal{F}$ by given condition.

Since $\mathcal{F} \subset \mathcal{M}$, then this implies that both A and X \ A are in $\mathcal{M}$. Hence $A \cap (X \setminus A) = \emptyset \in \mathcal{M}$, contradicting the fact that $\mathcal{M}$ is a filter.

2.4 Remark

If $\mathcal{U}$ is an ultrafilter on X, and $A \in \mathcal{U}$, then $\mathcal{U}$ contains all sets B with $A \subset B \subset X$. Indeed, if we start with such a B, then by the above result, either $B \in \mathcal{U}$ or $X \setminus B \in \mathcal{U}$.If $X \setminus B \in \mathcal{U}$, then $A \cap (X \setminus B) = \emptyset \in \mathcal{U}$, contradiction. Therefore B must belong to $\mathcal{U}$.

2.5 Corollary

The Fréchet filter $\mathcal{F}$on an infinite set$\mathbb{N}$ is not an ultrafilter.

Proof

Let $\mathbb{E}$ and $\mathbb{O}$ denote the sets of the even and odd numbers in $\mathbb{N}$ respectively.We known that $\mathbb{E} \cap \mathbb{O} = \emptyset$ and$\mathbb{E} \cup \mathbb{O} = \mathbb{N} \in \mathcal{F}$, but neither $\mathbb{E}$ nor $\mathbb{O}$ belongs to $\mathcal{F}$ because any set in $\mathcal{F}$has finite complement.

3. Types of Ultrafilters

There are two very different types of ultrafilters such as principal and non-principal (free).

3.1 Proposition

Any ultrafilter over a finite set is principal.

Proof

Let X be a finite set, $\mathcal{U}$ be an ultrafilter over P(X) and

$\mathcal{U} = \{S_1, S_2, \dots, S_k\}$.Since $\emptyset \notin \mathcal{U}$ and $S_i \cap S_j \in \mathcal{U}$ for every i, j,

$S_1 \cap S_2 \cap \dots \cap S_k \neq \emptyset$. If a $\in \cap_{i=1}^{k} S_i$, then a $\in \mathcal{U}$.But by the definition of principal ultrafilter$\{$ S : a $\in$ S$\}$ $\subset \mathcal{U}$.By the maximality of ultrafilter, $\mathcal{U}$ =$\{$S: a$\in$ S$\}$.

(or)

Let A be a finite set. Then either some a ∈ A satisfies {a} is in the ultrafilter, in which case it is principal; or else X \ {a} is in the ultrafilter for all a∈ A, so the finite intersection

$$A \cap (\cap_{a\in A} (X \setminus \{a\}) = A \cap (X \setminus \cup_{a\in A} \{a\}) = A \cap (X \setminus A) = \emptyset$$

is also in the ultrafilter.

So a non-principal ultrafilter must contain only infinite sets. In particular, if X is finite, then every ultrafilter on X is principal.

3.2 Proposition

Cofinite filter is intersection of all non-principal ultrafilters.

Proof

Let X be an infinite set.

Suppose that a set A⊆ X; we want to show that A is cofinite. Suppose for contradiction that A is not cofinite. That is, the set D= X \ A is infinite. From Proposition 3.1, the infinite set D belongs to some non-principal ultrafilter $\mathcal{U}$ on X. But $\mathcal{U}$ is a non-principal ultrafilter on X which does not contain A, contradicting our assumption that A belongs to every non-principal ultrafilter.

Let $\mathcal{F} = \{B \subseteq X: X \setminus B$ is finite$\}$, the cofinite filter on X. Then the collection $\{D\} \cup \mathcal{F}$ has the FIP, whence $\{D\} \cup \mathcal{F} \subset \mathcal{U}$ for some ultrafilter $\mathcal{U}$. Since $\mathcal{U}$ contains $\mathcal{F}$, it is non-principal.

3.3 Corollary

A non-principal ultrafilter is free.

Proof

If there exists $x \in \cap_{A \in \mathcal{F}} A$, then $X \setminus \{x\}$ is not an element of $\mathcal{F}$, by Theorem 2.3, $\{x\} \in \mathcal{F}$ and $\mathcal{F}$ is a principal ultrafilter.

3.4 Proposition

Every non-principal ultrafilter on an infinite set X contains the cofinite filter on X.

Proof

Let $\mathcal{U}$ be a non-principal ultrafilter on X and let $x \in X$ be arbitrary. Since $\mathcal{U}$ is an ultrafilter, exactly one of the sets $\{x\}$ and $X \setminus \{x\}$ belongs to $\mathcal{U}$, and since $\mathcal{U}$ is non-principal, $\{x\} \notin \mathcal{U}$. Thus, $X \setminus \{x\} \in \mathcal{U}$ for each $x \in X$. Now let F be any finite subset of X; then

$$X \setminus F = X \setminus \cup_{x \in F} \{x\} = \cap_{x \in \mathcal{F}} (X \setminus \{x\}) \in \mathcal{U}.$$

That is, $X \setminus F \in \mathcal{U}$. We have $\{X \setminus F: F \subseteq X$ is finite$\}$ is the cofinite filter on X.

Therefore $\mathcal{U}$ contains the cofinite filter.

3.5 Proposition

An ultrafilter on X is free if and only if it contains the Fréchet filter on X.

Proof

In the previous proposition we proved that every free ultrafilter on X contains the Fréchet filter on X. For the converse, suppose that $\mathcal{U}$ is a fixed (principal) ultrafilter on X; then there is an x∈ X such that {x} ∈ $\mathcal{U}$.

But X \ {x} is an element of the Fréchet filter that is not in $\mathcal{U}$, so $\mathcal{U}$ does not contain the Fre'chet filter.

4. Ultrafilter Convergence Theorem

4.1 Definition

A filter $\mathcal{F}$ on a topological space Y **converges** to a point y ∈ Y or y is a **limit** of $\mathcal{F}$ if for all open sets U containing y, U ∈ $\mathcal{F}$.

4.2 Theorem

Let Y be a topological space.

1. Y is Hausdorff if and only if every ultrafilter $\mathcal{F}$ on Y converges to at most one point.
2. Y is compact if and only if every ultrafilter $\mathcal{F}$ on Y converges to at least one point.

Proof

1. Suppose (to the contrary) that Y is Hausdorff, but x ≠ y are limit points of $\mathcal{F}$.

Since Y is Hausdorff, there exist disjoint open sets x ∈ U and y ∈V .By the definition of limit point, U, V ∈ $\mathcal{F}$ but U ∩ V= ∅, contradiction.

Conversely, suppose that Y is not Hausdorff. Then there are points x ≠ y such that every open neighborhood of x intersects every open neighborhood of y.

Then {U : x ∈ U open} ∪ { V : y ∈ V open } has the FIP. Let $\mathcal{F}$ be an ultrafilter containing it. So x and y are both limit points of $\mathcal{F}$.

2. Suppose to the contrary that Y is compact, but $\mathcal{F}$ has no limit points. Then for all $y \in Y$, there is an open set U_y containing y such that $U_y \notin \mathcal{F}$. So $Y = \cup_{y\in Y} U_y$ and by compactness, $Y = \cup_{i=1}^{n} U_{y_i}$. But $Y \in \mathcal{F}$, so some $U_{y_i} \in \mathcal{F}$, contradiction.

Conversely, suppose that Y is not compact. Then there is an open cover $Y = \cup_{y\in Y} U_y$ with no finite subcover. So $\cap_i (Y \setminus U_i) = \emptyset$, but no finite intersection is empty. Then $\{(Y \setminus U_i)\}_i$ has the FIP, so we can take an ultrafilter $\mathcal{F}$ containing it. Now for any point $y \in Y$, y is contained in some U_i, and $U_i \notin \mathcal{F}$, since $(Y \setminus U_i) \in \mathcal{F}$. So y is not a limit point of $\mathcal{F}$.

5.Comparison of filters on a set X

5.1 Definition

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two filters defined on a set X. We say that $\mathcal{F}_1$ is **finer** than $\mathcal{F}_2$ (or that $\mathcal{F}_2$ is **coarser** than $\mathcal{F}_1$) if $\mathcal{F}_2 \subset \mathcal{F}_1$.

5.2 Proposition

Let $(\mathcal{F}_i)_{i\in I}$ be a family of filters on a set X. Then $\mathcal{F} = \cap_{i\in I} \mathcal{F}_i$ is a filter on X and has the following properties.

(a) $\mathcal{F}$ is coarser than $\mathcal{F}_i (i \in I)$.

(b) If $\mathcal{F}'$ is a filter coarser than every $\mathcal{F}_i (i \in I)$ then $\mathcal{F}' \subset \mathcal{F}$.

Proof

For (F1), let $A \subset X$, $B \subset A$ and $B \in \mathcal{F}$. It follows that $B \in \mathcal{F}_i$ for every $i \in I$.

Since $\mathcal{F}_i$ is a filter and $B \subset A$, $A \in \mathcal{F}_i$ for every $i \in I$.

Thus $A \in \cap_{i\in I} \mathcal{F}_i = \mathcal{F}$.

For (F2), let $A_1, A_2, \dots, A_n \in \mathcal{F}$. For each j, $A_j \in \cap_{i\in I} \mathcal{F}_i$ and $A_j \in \mathcal{F}_i$ for every $i \in I$. Since $\mathcal{F}_i$ is a filter $(i \in I)$, $\cap_{j=1}^{n} A_j \in \mathcal{F}_i (i \in I)$. So $\cap_{j=1}^{n} A_j \in \cap_{i\in I} \mathcal{F}_i = \mathcal{F}$.

For (F3), for each i, $\mathcal{F}_i$ does not contain empty set, $\emptyset \notin \cap_{i \in I} \mathcal{F}_i$. Therefore $\mathcal{F}$ is a filter. Since $\mathcal{F} = \cap_{i \in I} \mathcal{F}_i$, $\mathcal{F} \subset \mathcal{F}_i (i \in I)$.That is, $\mathcal{F}$ is coarser than $\mathcal{F}_i (i \in I)$.

(a) Let $\mathcal{F}'$ be a filter coarser that $\mathcal{F}_i (i \in I)$. That is, $\mathcal{F}' \subset \mathcal{F}_i (i \in I)$.
Thus $\mathcal{F}' \subset \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$.

5.3 Definition

Let $(\mathcal{F}_i)_{i \in I}$ be a family of filters $\mathcal{F}_i$defined on set X. If there exists a filter $\mathcal{F}$ on X such that

(glb1)$\mathcal{F}$ is coarser than every $\mathcal{F}_i (i \in I)$.

(glb2) If $\mathcal{F}'$ is a filter on X such that $\mathcal{F}'$is coarser than every $\mathcal{F}_i (i \in I)$, then $\mathcal{F}' \subset \mathcal{F}$.Then $\mathcal{F}$ is called **the greatest lower bound** of the family $(\mathcal{F}_i)_{i \in I}$. Proposition (5.2) implies the greatest lower bound of a family $(\mathcal{F}_i)_{i \in I}$ of filters $\mathcal{F}_i$ on X always exists.

5.4 Definition

Let $(\mathcal{F}_i)_{i \in I}$ be a family of filters $\mathcal{F}_i$ on X. If there exists a filter $\bar{\mathcal{F}}$ on X such that

(lub1) $\bar{\mathcal{F}}$ is finer than every $\mathcal{F}_i (i \in I)$.

(lub2) If $\bar{\mathcal{F}}'$ is a filter on X such that $\bar{\mathcal{F}}'$is finer than every $\mathcal{F}_i, i \in I$, then $\bar{\mathcal{F}}$is called the **least upper bound** of $(\mathcal{F}_i)_{i \in I}$.

5.5 Proposition

Let $(\mathcal{F}_i)_{i \in I}$ be a family of filters on a set X. Then this family has a least upper bound in the set of all filters on X if and only if there exists a filter on X which is finer than every $\mathcal{F}_i$ for $i \in I$.

Proof (Necessary condition)

Assume that the least upper bound $\bar{\mathcal{F}}$ exists.(lub 1) implies $\bar{\mathcal{F}}$ is finer than every $\mathcal{F}_i$ for $i \in I$.

(Sufficient condition)

Assume that there exists a filter $\mathcal{F}$on X which is finer than $\mathcal{F}_i$ $(i \in I)$.Let Φ be the set of all filters which are finer than $\mathcal{F}_i$ $(i \in I)$.Then $\mathcal{F} \in \Phi$ and so Φ is non-empty.Let $\bar{\mathcal{F}}$ be the greatest lower bound of Φ.We prove that $\bar{\mathcal{F}}$ is least upper bound of $\mathcal{F}_i$ $(i \in I)$.

Let $\mathcal{F}_j \in (\mathcal{F}_i)_{i\in I}$. Since $\bar{\mathcal{F}}$ is the greatest lower bound of$\mathcal{F}_i$ $(i \in I)$, $\mathcal{F}_j \subset \bar{\mathcal{F}}$.

That is, $\bar{\mathcal{F}}$ is finer than every $\mathcal{F}_i$ $(i \in I)$.Put $g \in \Phi$. Then g is finer than every $\mathcal{F}_i$.Thusevery $\mathcal{F}_i$ is coarser than g of Φ. Hence $\mathcal{F}_i \subset \bar{\mathcal{F}}$.

Let $\mathcal{F}'$ be a filter on X such that$\mathcal{F}'$ is finer than every $\mathcal{F}_i$ $(i \in I)$.

Then $\mathcal{F}' \in \Phi$. Since $\bar{\mathcal{F}}$ be the greatest lower bound of Φ and $\mathcal{F}' \in \Phi$, (glb1) implies $\bar{\mathcal{F}} \subset \mathcal{F}'$.Therefore $\bar{\mathcal{F}}$ is the least upper bound of $(\mathcal{F}_i)_{i\in I}$.

6. Some Notions of Filter Basis and Trace of Filter

6.1 Definition

A collection $\mathfrak{B}$ of subsets of X is a **filter basis** if it satisfies the following two conditions:

(FB1) The intersection of two sets in $\mathfrak{B}$ contains a set of $\mathfrak{B}$.

(FB2) $\mathfrak{B}$ is non-empty and the empty subset of X does not belongs to $\mathfrak{B}$.

6.2 Definition

Let f: X⟶Y be a mapping from a set X into a set Y. Let $\mathfrak{B}$ be a filter basis on Y.Define $f^{-1}(\mathfrak{B})=\{ f^{-1}(A): A \in \mathfrak{B} \}$.

6.3 Proposition

Let $\mathfrak{B}$ be a filter basis on Y and f: X⟶Ybe a mapping. Then $f^{-1}(\mathfrak{B})$ is a filter basis on X if and only if $f^{-1}(A) \neq \emptyset$ for every $A \in \mathfrak{B}$.

Proof

Assume that $f^{-1}(\mathfrak{B})$ is a filter basis on X. (FB2) implies $f^{-1}(\mathfrak{B})$ is non-empty and empty subset of X does not belong to $f^{-1}(\mathfrak{B})$. For each $A \in \mathfrak{B}$, $f^{-1}(A) \in f^{-1}(\mathfrak{B})$.

So $f^{-1}(A) \neq \emptyset$.

Conversely, assume that $f^{-1}(A) \neq \emptyset$ for every $A \in \mathfrak{B}$. Since $\mathfrak{B}$ is a filter basis, $\mathfrak{B} \neq \emptyset$ and empty subset of Y does not belong to $\mathfrak{B}$. If $A \in \mathfrak{B}$, then $A \neq \emptyset$.

Moreover, $f^{-1}(A) \neq \emptyset$ for every $A \in \mathfrak{B}$.Therefore the empty subset of X does not belong to $f^{-1}(\mathfrak{B})$. Take $Z_1, Z_2 \in f^{-1}(\mathfrak{B})$.Then there exist $A_1, A_2 \in \mathfrak{B}$ such that $Z_1 = f^{-1}(A_1)$ and $Z_2 = f^{-1}(A_2)$. If $A_1, A_2 \in \mathfrak{B}$, then there exists $A_3 \in \mathfrak{B}$ such that $A_3 \subset A_1 \cap A_2$. It follows that $f^{-1}(A_3) \subset f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2) = Z_1 \cap Z_2$.Therefore $f^{-1}(\mathfrak{B})$ is a filter basis on X, if $f^{-1}(A) \neq \emptyset$ for every $A \in \mathfrak{B}$.

6.4 Definition

Let A be a non-empty subset of a set X and $\mathcal{F}$ a filter on X. Then the **trace** of $\mathcal{F}$ on A is defined and denoted by $\mathcal{F}_A = \{A \cap B : B \in \mathcal{F}\}$.

6.5 Proposition

If $\mathfrak{B}$ is a filter basis on X, then the trace $\mathfrak{B}_A = \{A \cap B : B \in \mathfrak{B}\}$ is a filter basis on A if and only if $A \cap B \neq \emptyset$ for every $B \in \mathfrak{B}$.

Proof

Let $f: A \to X$ be the canonical injection of A into X defined by $f(x) = x$.

Let $B \in \mathfrak{B}$.

$f^{-1}(B) = \{x \in A : f(x) \in B\} = \{x \in A : x \in B\} = A \cap B$.

$f^{-1}(\mathfrak{B}) = \{f^{-1}(B) : B \in \mathfrak{B}\} = \{A \cap B : B \in \mathfrak{B}\} = \mathfrak{B}_A$.

Proposition 6.3 implies $f^{-1}(\mathfrak{B})$ is a filter basis if and only if $f^{-1}(B) \neq \emptyset$ for every $B \in \mathfrak{B}$. Therefore $\mathfrak{B}_A$ is a filter basis if and only if $A \cap B \neq \emptyset$ for every $B \in \mathfrak{B}$.

7. Convergence of Cauchy Filter

7.1 Definition

Let X be a topological space and $\mathfrak{B}$ a filter basis on X. A point x of X is said to **adhere** to $\mathfrak{B}$ if x adheres to every set A in $\mathfrak{B}$.

7.2 Definition

Let E be a topological vector space and $A \subset E$. A filter $\mathcal{F}$ on A is said to be a **Cauchy filter** if for every neighborhood of zero V, there exists a set $X \in \mathcal{F}$ such that $X - X \subset V$.

7.3 Proposition

Suppose that $\mathcal{F}$ is a filter on a set A of a topological vector space E and that $\mathcal{F}$ converges to a point $x \in E$. Then $\mathcal{F}$ is a Cauchy filter on A.

Proof

Assume that $\mathcal{F}$ on A converges to $x \in E$. Let V be neighborhood of zero in E. Then there exists a balanced neighborhood U of zero such that $U + U \subset V$. Since $\mathcal{F}$ converges to x, $\mathcal{B}(x) \subset \mathcal{F}$. Thus $x + U \in \mathcal{B}(x) \subset \mathcal{F}$. Then there exists $X \in \mathcal{F}$ such that $X \subset x + U$.

Let $z \in X - X$. Then there exists y, $w \in X$ such that $z = y - w$. Since y, $w \in X$ and $X \subset x + U$, $y - x$ and $w - x \in U$. Since U is balanced, $w - x \in U$ implies $x - w \in U$.

Thus $(y - x) + (x - w) \in U + U \subset V$. So $z = y - w \in V$, for every $z \in X - X$. Hence $X - X \subset V$. Therefore $\mathcal{F}$ is a Cauchy filter.

7.4 Proposition

If the point x adheres to the Cauchy filter $\mathcal{F}$ on a set A of topological vector space E, then $\mathcal{F}$ converges to x.

Proof

Let $\mathcal{F}$ be a Cauchy filter on A and x adheres to $\mathcal{F}$.

Take W ∈ $\mathcal{B}$(x), where $\mathcal{B}$(x) is the set of neighborhood of x.

Hence there exists a neighborhood V of zero such that x +V ⊂ W.

Therefore there exists a neighborhood U of zero such that U+U ⊂V. Since $\mathcal{F}$ is Cauchy filter, there exists X ∈ $\mathcal{F}$ such that X−X ⊂U. x∈ $\overline{X}$ since x adheres to X and x + U is a neighborhood of x and hence

(x+ U) ∩ X≠ ∅.

Take y ∈ (x + U) ∩X .Then y ∈ x + U and y ∈ X.

Let z ∈ X. Then z −y ∈ X−X ⊂U.So z ∈ y+ U ⊂ x + U +U⊂ x + V⊂ W. Hence X ⊂ W.Since X ∈ $\mathcal{F}$ and X ⊂W, W∈ $\mathcal{F}$ and $\mathcal{B}$(x) ⊂ $\mathcal{F}$.

Therefore $\mathcal{F}$ converges to x.

Acknowledgements

I am very grateful to the Myanmar Academy of Arts and Science for allowing the presentation of this research paper. I would like to appreciate Dr Win Naing, Rector, Dr Sein Sein Aung and DrThetNaingOo, Pro-Rectors of Monywa University, for their suggestions. I am also very thankful to Dr Zaw Myint, Professor and Head, Department of Mathematics, Monywa University, for his valuable useful comments.

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